

The R_0 -type fuzzy logic metric space and an algorithm for solving fuzzy modus ponens

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Abstract

Fuzzy modus ponens (briefly, FMP) is the most fundamental form of fuzzy reasoning and has been extensively discussed by diverse researchers. The aim of the present paper is to propose a formalized form of FMP, called generalized modus ponens, in the fuzzy logic system L^* and solve it in L^* , and then provide its numerical version as a new algorithm for solving FMP. As a preparation, some related questions such as what is a fuzzy logic metric space, why is R_0 -implication operator selected, and what kind of merits does the fuzzy pseudo-metric logic space $(F(S), \rho)$ possess, etc. are analyzed.

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1. Introduction

At first glance, the methodology of fuzzy reasoning differs from the methodology of artificial intelligence. In fact, as pointed out by Dubois and Prade a decade ago in their survey paper [1] that “Zadeh’s approximate reasoning methodology was devised outside the powerful stream of thought that emerged under the name ‘artificial intelligence’ while there is obviously a close relationship between both. The main reason for this *gap* seems to be that, from the beginning artificial intelligence emphasized symbolic manipulation and has rooted in logic, automated deduction using syntactic tools, and has very much neglected anything pertaining to ‘number crunching’. On the contrary, Zadeh’s methodology was right away addressing the interface between numbers and symbols, by proposing a reasoning methodology based on non-linear optimization.” In recent years, this situation seems to have changed, and many research papers as well as monographs brought to light the above mentioned *gap* which is now vanishing gradually (see, e.g., [2–6]), and Zadeh pointed out in [7] that “as a label, fuzzy logic, FL, has two different meanings. More specifically, in a narrow sense, fuzzy logic, FL_n, is a logic system which aims at a formalization of approximate reasoning.” “In a wide sense, fuzzy logic, FL_w, is coextensive with fuzzy set theory, FST. FL_w is far broader than FL_n

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and contains FLn as one of its branches.” Hence we see that FLn is closely related to artificial intelligence (briefly, AI), and the above mentioned *gap* between FLn and AI has been greatly reduced. It seems that Hajek’s book [2] is one of the standard monograph discussing FLn in a formalized way, the way of AI, where the above mentioned *gap* no longer exists. The present paper also aims at FLn while formalized deduction and numerical computation are employed cooperatively. We first propose a formalized form of fuzzy modus ponens (briefly, FMP) and then provide a new algorithm for solving FMP. The formal version is given and solved in the fuzzy logic system L^* proposed by the first author (see, [3,8]), the corresponding numerical version of it is a new algorithm for solving FMP which can be used to simplify the computation of the fuzzy conclusion of FMP whenever the implication operator R_0 (see below) is employed. Because the R_0 -type fuzzy logic metric space $(F(S), \rho)$ plays the leading role in the present paper, some basic concepts and related questions should be clarified in detail in advance, and hence the present paper consists of the following sections:

- What is $(F(S), \rho)$?
- Why a fuzzy pseudo-metric?
- Why the R_0 -implication operator?
- What is the importance in justifying that there are no isolated points in the R_0 -type fuzzy logic metric space $(F(S), \rho)$, and that the basic operators on $(F(S), \rho)$ are continuous?
- Continuity of basic logic connectives in the R_0 -type fuzzy logic metric space $(F(S), \rho)$.
- An algorithm for FMP and its application.
- Conclusion.

2. What is $(F(S), \rho)$?

The basic concept of an abstract proposition in classical mathematical logic is defined as follows. First let $S = \{p_1, p_2, \dots\}$ be a countable set of abstract symbols p_1, p_2, \dots , called atomic propositions (or atomic formulas), and $F(S)$ be the free algebra of type $(\neg, \vee, \rightarrow)$ generated by S , i.e., $S \subset F(S)$ and $F(S)$ is the smallest set closed under the operations \neg, \vee and \rightarrow , where \neg is a unary operation, \vee and \rightarrow are binary operations, and $A \vee B$ is an abbreviation of $\neg A \rightarrow B$ ($A, B \in F(S)$) and hence the logic connective “ \vee ” may or may not appear in the above definition of $F(S)$. Members of $F(S)$ are well-formed formulas (or simply, formulas or, abstract propositions). We define on $\{0, 1\}$ \neg, \vee , and \rightarrow by letting $\neg 0 = 1, \neg 1 = 0, a \vee b = \max\{a, b\}$ and $a \rightarrow b = 0$ if and only if $a = 1$ and $b = 0$, then both $F(S)$ and $\{0, 1\}$ are algebras with the same type $(\neg, \vee, \rightarrow)$. Then a valuation v of $F(S)$ is a mapping $v : F(S) \rightarrow \{0, 1\}$ preserving the operations \neg, \vee , and \rightarrow , i.e., $v(\neg A) = \neg v(A)$, $v(A \vee B) = v(A) \vee v(B)$, and $v(A \rightarrow B) = v(A) \rightarrow v(B)$. The set of all valuations of $F(S)$ is denoted by Ω . Let A be a proposition of $F(S)$, A is said to be a tautology if $\forall v \in \Omega, v(A) = 1$ holds, A is said to be a contradiction if $\forall v \in \Omega, v(A) = 0$ holds. This is the elementary semantics of classical propositional logic (see [9]). As for formalized Fuzzy logic, only two things are modified: (i) The valuation field should be expanded from $\{0, 1\}$ to $[0, 1]$ (like the fact that a crisp subset A of U can be considered to be a mapping $A : U \rightarrow \{0, 1\}$ and a fuzzy subset \tilde{A} can be considered to be a mapping $\tilde{A} : U \rightarrow [0, 1]$). (ii) More logic connectives have to be taken into consideration and, especially, the disjunction \vee has to be taken into account because the relations among \vee, \neg , and \rightarrow possess different forms in different fuzzy logic systems not like the situation of 2-valued classical logic where $A \vee B$ is an abbreviation of $\neg A \rightarrow B$. Notice that each formula of $F(S)$ induces a function in a natural way (see [10]): let $A(p_1, \dots, p_n)$ be a formula obtained by connecting the atomic formulas p_1, \dots, p_n with the logic connectives \neg, \vee , and \rightarrow , then a corresponding function $\bar{A} : [0, 1]^n \rightarrow [0, 1]$ can be obtained by substituting p_i by x_i in $A(p_1, \dots, p_n)$ ($i = 1, \dots, n$). For example, if $A = (p_1 \rightarrow \neg p_2) \vee p_3$, then $\bar{A}(x_1, x_2, x_3) = (x_1 \rightarrow \neg x_2) \vee x_3 = \max\{x_1 \rightarrow (1 - x_2), x_3\} = \max\{R(x_1, 1 - x_2), x_3\}$, where $\neg x_2 = 1 - x_2$ is assumed. As for R , one can choose the Lukasiewicz implication operator, Gödel implication operator, or R_0 -implication operator. These operators will be discussed later. The concept of resemblance degree has been used many times before (see, e.g., [11]), but we define resemblance degree between two formulas A and B by means of an integral in [10] as follows:

$$\xi_R(A, B) = \int_{\Delta} R(\bar{A}, \bar{B}) \wedge R(\bar{B}, \bar{A}) dw, \quad (1)$$

where A and B contain n -atomic formulas and $\Delta = [0, 1]^n$. where, without any loss of generality, we assumed that both A and B contain one and the same group of atomic formulas, say, p_1, \dots, p_n . For example,

if $A = p_1 \rightarrow p_2, B = p_2 \vee p_3 \rightarrow p_4$, then A and B can be re-written as their logical equivalent forms $A_1 = (p_1 \rightarrow p_2) \wedge (p_3 \rightarrow p_3) \wedge (p_4 \rightarrow p_4)$ and $B_1 = (p_1 \rightarrow p_1) \wedge (p_2 \vee p_3 \rightarrow p_4)$ respectively and $\xi_R(A_1, B_1) = \xi_R(A, B)$ holds. Then let

$$\rho_R(A, B) = 1 - \xi_R(A, B), \quad A, B \in F(S), \quad (2)$$

it is pointed out in [10] that ρ_R is a fuzzy pseudo-metric on $F(S)$, i.e., a pseudo-metric with values varying in the unit interval $[0, 1]$, and $(F(S), \rho)$ is called a fuzzy logic metric space. ρ_R can also be simplified to be ρ if no confusion arises. It is in this way that we obtained the pseudo-metric space $(F(S), \rho)$. It is clear that if $A \approx B$, i.e., A and B are logically equivalent, then $\rho(A, B) = 0$, but not vice versa when R is not continuous.

3. Why a fuzzy pseudo-metric?

As was pointed out in [12], “the theory of fuzzy binary relations is probably one of the most important and influential branches of fuzzy set theory”, and the concept of similarity relations proposed by Zadeh in [13] is an important special fuzzy binary relation and has been thoroughly investigated. Fuzzy similarity relation can be used even in classical predicate logic to establish approximate reasoning theory (see [14]). On the other hand, the concept of fuzzy similarity relations on a set X had been greatly generalized by many scholars several decades ago where the Min-operation \wedge in the “transitive condition” of a fuzzy similarity relation had been changed to be a general t -norm $*$ and the fuzzy relation obtained was called a Fuzzy Equivalence Relation (briefly, FER) in the survey paper [15]. In fact, an FER E on X is a binary relation $E : X \times X \rightarrow [0, 1]$ satisfying the conditions that

$$E(x, x) = 1, \quad (3)$$

$$E(x, y) = E(y, x), \quad (4)$$

$$E(x, y) * E(y, z) \leq E(x, z), \quad (5)$$

where $*$ is a t -norm. Assume in the following that $*$ is the Lukasiewicz t -norm, i.e., $a * b = (a + b - 1) \vee 0$. Let $\rho(x, y) = 1 - E(x, y) (x, y \in X)$, then it is clear that

$$\rho(x, x) = 0, \quad (3')$$

$$\rho(x, y) = \rho(y, x), \quad (4')$$

and it follows from (5) that

$$\begin{aligned} \rho(x, y) + \rho(y, z) &= (1 - E(x, y)) + (1 - E(y, z)) \\ &= 1 - (E(x, y) + E(y, z) - 1) \vee 0 \\ &= 1 - E(x, y) * E(y, z) \\ &\geq 1 - E(x, z) = \rho(x, z), \end{aligned}$$

hence

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z). \quad (5')$$

We see from (3')–(5') that $\rho : X \times X \rightarrow [0, 1]$ is a fuzzy pseudo-metric on X . Conversely, let ρ be a fuzzy pseudo-metric on X and $*$ be the Lukasiewicz t -norm. Define a binary relation E on X by letting $E(x, y) = 1 - \rho(x, y)$, then it is clear that $E(x, x) = 1$, $E(x, y) = E(y, x)$, and it follows from (5') that

$$\begin{aligned} E(x, y) * E(y, z) &= (E(x, y) + E(y, z) - 1) \vee 0 = (1 - \rho(x, y) + 1 - \rho(y, z) - 1) \vee 0 \\ &= (1 - (\rho(x, y) + \rho(y, z))) \vee 0 \leq (1 - \rho(x, z)) \vee 0 \leq E(x, z). \end{aligned} \quad (6)$$

Hence E is an FER on X .

From the above discussion we see that a fuzzy pseudo-metric on X is equivalent to a special FER on X when the Lukasiewicz t -norm is employed. By a fuzzy similarity relation on X one can measure to what extent two elements x and y of X are close to each other, and a fuzzy pseudo-metric ρ on X can play the same role, i.e. the smaller $\rho(x, y)$ is, the closer x and y are. In the present paper, a fuzzy pseudo-metric will be employed throughout. This is because,

for example, once a pseudo-metric ρ is given on X , then a convergence theory appears on (X, ρ) , assume that $p \in X$ and p is not an isolated point, then one can construct a sequence x_1, x_2, \dots to approximate it in a natural way. What is more, assume that $f : X^n \rightarrow [0, 1]$ is a function, then one can discuss continuity of f on X in an intimate way well-known in analysis. Especially, when X is the set $F(S)$ consisting of all propositions of a certain fuzzy logic system, then one can establish an approximate reasoning theory on $(F(S), \rho)$ (see, for example, [10,16]). In the present paper a special fuzzy pseudo-metric will be employed throughout which can be naturally induced by the R_0 -implication operator and possesses good properties as will be shown subsequently.

4. Why the R_0 -implication operator?

An implication operator R on $[0, 1]$ is a function $R : [0, 1]^2 \rightarrow [0, 1]$ satisfying certain conditions. Different groups of conditions were requested by different authors (see, for example, [1,17]). Ten conditions were listed and analyzed in [1], and many interesting properties of implication operators over T - and S -norms were discussed in detail in [17]. Because the present paper aims to follow the formalized way, in what follows, an implication operator R is requested to fulfill the condition that there exists a left-continuous T -norm \otimes on $[0, 1]$ such that

$$x \otimes y \leq z \quad \text{if and only if} \quad x \leq R(y, z), \quad (7)$$

where a T -norm \otimes is said to be left-continuous (briefly, LC), if

$$x \otimes \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \otimes y_i), \quad x, y_i \in [0, 1], i \in I. \quad (8)$$

An implication operation R is said to be residual if there exists a T -norm \otimes such that (7) holds. If \otimes is left-continuous (not necessarily continuous), then the corresponding residual implication operation R possesses satisfactory properties (see, [8]).

Left-continuous T -norms and related logic systems have been thoroughly investigated (see, for example, [18]). An implication operator R accompanied by an LC T -norm \otimes satisfying (7) is called a *regular implication operator* w.r.t. \otimes by the first author in [8]. It seems that the Lukasiewicz implication operator R_L is the most famous regular implication operator w.r.t. \otimes where

$$R_L(a, b) = (1 - a + b) \wedge 1, \quad (9)$$

and the accompanied T -norm \otimes is defined by

$$a \otimes b = (a + b - 1) \vee 0. \quad (10)$$

Notice that the Lukasiewicz implication operator R_L (briefly, \rightarrow) satisfies the condition of NM algebra (see [18]) as follows

$$(a \otimes b \rightarrow 0) \vee (a \wedge b \rightarrow a \otimes b) = 1, \quad (a \rightarrow 0) \rightarrow 0 = a. \quad (11)$$

The first author introduced in [3] a regular implication operator, called R_0 -implication, of which the pair (R_0, \otimes_0) is defined by the following formulas

$$R_0(a, b) = \begin{cases} 1, & a \leq b, \\ (1 - a) \vee b, & a > b. \end{cases} \quad a \otimes_0 b = \begin{cases} a \wedge b, & a + b > 1, \\ 0, & a + b \leq 1. \end{cases} \quad (12)$$

It is obvious that R_0 satisfies the NM condition (11) and therefore the R_0 -implication operator can be axiomatized in the standard sense. In fact, it is proved in [19] that the logic system L^* corresponding to R_0 -implication is equivalent to the logic system NM (see [20]), hence a logic formula is a theorem in L^* iff it is a 1-tautology w.r.t. the R_0 interval $[0, 1]$. This fact can be used to characterize logic properties by means of integrated semantics (see [10]). Both R_L and R_0 own this kind of benefit. For example, the integrated truth degree of a logic formula A equals 1 iff A is an almost tautology in the logic system L^* (see [21]). In consideration of the following fact the present paper chooses and employs the R_0 -implication operator: Suppose that a group Γ of propositions are given, say $\Gamma = \{A, B\}$, and C is any proposition, if we are asked to evaluate to what extent C is a conclusion of Γ , i.e., to what extent $\Gamma \vdash C$ is true, then there is an easy way in the system L^* for answering this question: because in the system L^* we have (see [8])

$$\{A, B\} \vdash C \quad \text{if and only if} \quad \vdash A^2 \rightarrow (B^2 \rightarrow C), \quad (13)$$

where $A^2 = A \otimes A = \neg(A \rightarrow \neg A)$, $B^2 = \neg(B \rightarrow \neg B)$, hence one can answer the question by saying that to an extent αC is a conclusion of $\{A, B\}$, where α is the integrated truth degree (see [10]) of $A^2 \rightarrow (B^2 \rightarrow C)$. On the contrary, this benefit vanishes in the system Luk, because we have in Luk that $\{A, B\} \vdash C$ if and only if there exist natural numbers n_1 and n_2 such that $\vdash A_1^{n_1} \rightarrow (A_2^{n_2} \rightarrow C)$ holds (see [2]) and hence we cannot calculate the integrated truth degree of $A_1^{n_1} \rightarrow (A_2^{n_2} \rightarrow C)$ because it contains two undecided numbers n_1 and n_2 .

From the above analysis we see that the R_0 -implication operator and the corresponding fuzzy logic system L^* seem to be, in a certain sense, convenient for use.

5. What is the importance in justifying that there are no isolated points in $(F(S), \rho)$, and that the basic operators on $(F(S), \rho)$ are continuous?

The essentials of fuzzy reasoning is approximate rather than precise, and this is true especially for the formalized version of fuzzy reasoning. As mentioned above, an easy and convenient way for characterizing approximation is to construct a pseudo-metric ρ on the set of formal fuzzy propositions as has been done in [8,10,16]. And once the logic metric space $(F(S), \rho)$ is constructed and we are to establish an approximate reasoning theory on $(F(S), \rho)$, two important problems have to be clarified:

1° Does $(F(S), \rho)$ contain isolated point?

2° If or not the basic operators \neg , \vee , and \rightarrow are continuous with respect to ρ ?

For, if $(F(S), \rho)$ contains an isolated point, say, B , then there would be no other propositions around B in a small enough neighborhood $N(B, \varepsilon)$ of B , and hence one could not discuss rules like “if A is very close to B , then ...” because there is no A which is very close to B . Second, one certainly hopes that $A_n \rightarrow B_n$ is close to $A \rightarrow B$ whenever A_n is close to A and B_n is close to B , i.e., one hopes that the implication operator \rightarrow is continuous with respect to ρ , because otherwise it would be difficult to establish a reasonable approximate reasoning theory in $(F(S), \rho)$. Fortunately, we will see below that there are no isolated points in $(F(S), \rho)$ where $\rho = \rho_{R_0}$ and, it seems to be a surprise, the implication operator \rightarrow on $F(S)$ induced by the discontinuous R_0 -implication operator $R_0 : [0, 1]^2 \rightarrow [0, 1]$ is continuous with respect to ρ , where $\rho = \rho_{R_0}$ is defined by (2). As for the operators \neg and \vee , it is easier to verify their continuity on $(F(S), \rho)$.

6. There is no isolated point in the R_0 -type logic metric space $(F(S), \rho)$

Theorem 1. Suppose that $\rho = \rho_{R_0}$ is defined by (1) and (2), then there is no isolated point in the R_0 -type logic metric space $(F(S), \rho)$.

The following lemma given in [10] is necessary for proving Theorem 1:

Lemma 1. Let $I_n = p_1 \wedge \cdots \wedge p_n$ and $U_n = p_1 \vee \cdots \vee p_n$, where p_1, \dots, p_n are different atomic formulas of S , then

$$\tau_R(I_n) = \int_{\Delta_n} \bar{I}_n dw_n = \underbrace{\int_0^1 \cdots \int_0^1}_{n} (x_1 \wedge \cdots \wedge x_n) dx_1 \cdots dx_n = \frac{1}{n+1}, \quad (14)$$

$$\tau_R(U_n) = \int_{\Delta_n} \bar{U}_n dw_n = \underbrace{\int_0^1 \cdots \int_0^1}_{n} (x_1 \vee \cdots \vee x_n) dx_1 \cdots dx_n = \frac{n}{n+1}, \quad (15)$$

where R is any implication operator.

Clearly, let $I_n^* = p_{n+1} \wedge \cdots \wedge p_{2n}$ and $U_n^* = p_{n+1} \vee \cdots \vee p_{2n}$, then we still have $\tau_R(I_n^*) = \frac{1}{n+1}$ and $\tau_R(U_n^*) = \frac{n}{n+1}$.

Proof of Theorem 1. Recall that the R -integrated truth degree $\tau_R(A)$ of an abstract proposition A is defined by

$$\tau_R(A) = \int_{\Delta} \bar{A} dw, \quad (16)$$

where R is an implication operator, $A = A(p_1, \dots, p_n)$ contains n -atomic formulas and $\bar{A} = \bar{A}(x_1, \dots, x_n)$ is the A -induced function, and $\Delta = [0, 1]^n$, $dw = dx_1 \cdots dx_n$ (see [10]). Assume that ε is any given number in $[0, 1]$, it is only necessary to find a formula B in $F(S)$ such that

$$0 < \rho(A, B) < \varepsilon, \quad (17)$$

where $\rho = \rho_{R_0}$. In fact, choose n large enough such that $\frac{1}{n+1} < \varepsilon$. If $\tau_{R_0}(A) = 1$, then $\bar{A}(x_1, \dots, x_n) = 1$ holds almost everywhere in $[0, 1]^n$. Let $B = U_n$, notice that $R_0(1, u) = u$ and $R_0(u, 1) = 1$ we see from (1) that

$$\xi_{R_0}(A, B) = \int_{\Delta} R_0(\bar{A}, \bar{U}_n) \wedge R_0(\bar{U}_n, \bar{A}) dw = \int_{\Delta} \bar{U}_n dw = \frac{n}{n+1}. \quad (18)$$

Hence it follows from (18) and (2) that (17) holds. If $\tau_{R_0}(A) = 0$, let $B = I_n$, then it can be proved similarly that (17) holds. Lastly, assume that $0 < \tau_{R_0}(A) < 1$. Choose n large enough such that $\frac{1}{n+1} < \varepsilon$ and let $B = A \vee I_n^*$. Notice that (see [8,16])

$$R_0(a \vee b, c) = R_0(a, c) \wedge R_0(b, c), \quad (19)$$

$$R_0(a, b) = 1 \quad \text{if and only if} \quad a \leq b, \quad (20)$$

and

$$b \leq c \quad \text{implies that} \quad R_0(a, b) \leq R_0(a, c), \quad (21)$$

we have from $\bar{A} \leq \bar{B}$ that

$$\begin{aligned} \xi_{R_0}(A, B) &= \int_{\Delta} R_0(\bar{A}, \bar{B}) \wedge R_0(\bar{B}, \bar{A}) dw = \int_{\Delta} R_0(\bar{B}, \bar{A}) dw \\ &= \int_{\Delta} R_0(\bar{A} \vee \bar{I}_n^*, \bar{A}) dw = \int_{\Delta} R_0(\bar{A}, \bar{A}) \wedge R_0(\bar{I}_n^*, \bar{A}) dw \\ &= \int_{\Delta} R_0(\bar{I}_n^*, \bar{A}) dw \geq \int_{\Delta} R_0(\bar{I}_n^*, 0) dw = \int_{\Delta} \bar{U}_n^* dw = \frac{n}{n+1}. \end{aligned}$$

Hence we have from (2) that $\rho(A, B) \leq 1 - \frac{n}{n+1} = \frac{1}{n+1}$. Moreover, it can be proved from the assumption that A and I_n^* contain no atomic formula in common such that $\rho(A, B) > 0$. Therefore (17) holds. This completes the proof. \square

7. Continuity of basic logic connectives in the R_0 -type logic metric space $(F(S), \rho)$

Continuities of the negation operation \neg and the disjunction operation \vee are easy to prove, in the following we only prove continuity of the implication operation \rightarrow with respect to ρ , where $\rho = \rho_{R_0}$ is defined by (1) and (2). The following lemmas are needed.

Lemma 2. Let $f : \Delta \rightarrow [0, 1]$ be a measurable function where $\Delta = [0, 1]^n$, and $\varepsilon \in (0, 1)$. If

$$\int_{\Delta} f dw > 1 - \varepsilon^2, \quad (22)$$

then Δ has a measurable subset E such that $\mu(E) < \varepsilon$ and $f(X) \geq 1 - \varepsilon$ holds whenever $X \in \Delta - E$.

Proof. Let $H = \{X \in \Delta \mid f(X) \geq 1 - \varepsilon\}$, $E = \Delta - H$, it only needs to be proved that $\mu(E) < \varepsilon$. In fact, suppose on the contrary $\mu(E) \geq \varepsilon$, then it follows from $f(X) < 1 - \varepsilon$ ($X \in E$) that

$$\int_E f dw \leq (1 - \varepsilon)\mu(E),$$

hence

$$\begin{aligned} \int_{\Delta} f dw &= \int_E f dw + \int_{\Delta - E} f dw \leq (1 - \varepsilon)\mu(E) + \mu(\Delta - E) \\ &= (1 - \varepsilon)\mu(E) + (1 - \mu(E)) = 1 - \varepsilon\mu(E) \leq 1 - \varepsilon^2. \end{aligned}$$

This contradicts (22). \square

Lemma 3. Suppose that $R_0 : [0, 1]^2 \rightarrow [0, 1]$ is the R_0 -implication operator, write $a \rightarrow b$ for $R_0(a, b)$, if

$$(a \rightarrow b) \wedge (b \rightarrow a) > 1 - \varepsilon, \quad \varepsilon \in (0, 1), \quad (23)$$

then $\{a, b\} \subset [0, \varepsilon]$, or $\{a, b\} \subset (1 - \varepsilon, 1]$ whenever $a \neq b$.

Proof. Assume that $a < b$, then it follows from (12) that $a \rightarrow b = 1$ and we have from (12) and (23) that

$$b' \vee a > 1 - \varepsilon, \quad b' = 1 - b, \quad b \in [0, 1]. \quad (24)$$

If $a \geq b'$, then it follows from (24) that $a > 1 - \varepsilon$ and $b > a > 1 - \varepsilon$, hence $\{a, b\} \subset (1 - \varepsilon, 1]$. If $a < b'$, then it follows from $b' > 1 - \varepsilon$ that $b < \varepsilon$ and $a < b < \varepsilon$, hence $\{a, b\} \subset [0, \varepsilon]$. The assertion of Lemma 3 can be proved similarly for the case $a > b$. \square

Lemma 4. Let $(F(S), \rho)$ be the R_0 -type fuzzy logic metric space where $\rho = \rho_{R_0}$. If $\lim_{n \rightarrow \infty} A_n = A$, then

$$\lim_{n \rightarrow \infty} (A_n \rightarrow B) = A \rightarrow B, \quad A, B, A_n \in F(S). \quad (25)$$

Proof. Consider the integrated resemblance degree (see [10]) between $A_n \rightarrow B$ and $A \rightarrow B$

$$\xi(A_n \rightarrow B, A \rightarrow B) = \int_{\Delta} [(\bar{A}_n \rightarrow \bar{B}) \rightarrow (\bar{A} \rightarrow \bar{B})] \wedge [(\bar{A} \rightarrow \bar{B}) \rightarrow (\bar{A}_n \rightarrow \bar{B})] dw. \quad (26)$$

It only needs to be proved that $\forall \varepsilon \in (0, 1)$, there exists a positive number M such that $\xi(A_n \rightarrow B, A \rightarrow B) > (1 - \varepsilon)^2$ whenever $n \geq M$.

In fact, it follows from $\lim_{n \rightarrow \infty} A_n = A$ that there exists a natural number M such that $\rho(A_n, A) < \varepsilon^2$ whenever $n \geq M$, i.e.,

$$\xi(A_n, A) = \int_{\Delta} (\bar{A}_n \rightarrow \bar{A}) \wedge (\bar{A} \rightarrow \bar{A}_n) dw > 1 - \varepsilon^2. \quad (27)$$

Since $f = (\bar{A}_n \rightarrow \bar{A}) \wedge (\bar{A} \rightarrow \bar{A}_n)$ is measurable (see [22]) it follows from Lemma 2 that Δ has a measurable subset E such that $\mu(E) < \varepsilon$ and

$$(\bar{A}_n \rightarrow \bar{A}) \wedge (\bar{A} \rightarrow \bar{A}_n) > 1 - \varepsilon \quad (28)$$

holds for $\Delta - E$. If $\bar{A}_n(w) \neq \bar{A}(w)$, then it follows from Lemma 3 that $\{\bar{A}_n(w), \bar{A}(w)\} \subset [0, \varepsilon]$ or $\{\bar{A}_n(w), \bar{A}(w)\} \subset (1 - \varepsilon, 1]$. We only consider the case of $\{\bar{A}_n(w), \bar{A}(w)\} \subset [0, \varepsilon]$, then $\bar{A}_n(w) < \varepsilon, \bar{A}(w) < \varepsilon$. Let f be the integrand function in (26), we are to prove that

$$f(w) \geq 1 - \varepsilon, \quad w \in \Delta - E, \quad \text{and} \quad \bar{A}_n(w) \neq \bar{A}(w), \quad (29)$$

because then by letting $H = \{w \in \Delta - E | \bar{A}_n(w) = \bar{A}(w)\}$, $G = \Delta - E - H$ we will have

$$\begin{aligned} \xi(A_n \rightarrow B, A \rightarrow B) &\geq \int_{\Delta - E} f(w) dw = \int_H f(w) dw + \int_G f(w) dw = \int_H 1 \cdot dw + \int_G f(w) dw \\ &\geq \mu(H) + (1 - \varepsilon)\mu(G) \geq (1 - \varepsilon)(\mu(H) + \mu(G)) = (1 - \varepsilon)\mu(\Delta - E) \geq (1 - \varepsilon)^2 \end{aligned}$$

and the proof will be completed. \square

In fact, it follows from $a \rightarrow b \geq b$, $(a \rightarrow c) \wedge (b \rightarrow c) = a \vee b \rightarrow c$, $a \rightarrow b \geq a' \vee b \geq a'$, and $(a \vee b)' = a' \wedge b'$ that

$$\begin{aligned} f(w) &= [(\bar{A}_n(w) \rightarrow \bar{B}(w)) \rightarrow (\bar{A}(w) \rightarrow \bar{B}(w))] \wedge [(\bar{A}(w) \rightarrow \bar{B}(w)) \rightarrow (\bar{A}_n(w) \rightarrow \bar{B}(w))] \\ &\geq (\bar{A}(w) \rightarrow \bar{B}(w)) \wedge (\bar{A}_n(w) \rightarrow \bar{B}(w)) = \bar{A}(w) \vee \bar{A}_n(w) \rightarrow \bar{B}(w) \\ &\geq (\bar{A}(w) \vee \bar{A}_n(w))' = \bar{A}'(w) \wedge \bar{A}_n'(w). \end{aligned} \quad (30)$$

Since $\bar{A}(w) < \varepsilon, \bar{A}_n(w) < \varepsilon$, it follows from (30) that $f(w) > 1 - \varepsilon$. This proves (29).

The proof for the case of $\{\bar{A}_n(w), \bar{A}(w)\} \subset (1 - \varepsilon, 1]$ can be similarly obtained.

Theorem 2. The implication operator \rightarrow is continuous in the R_0 -type fuzzy logic metric space $(F(S), \rho)$.

Proof. First notice that if $\lim_{n \rightarrow \infty} B_n = B$, then $\lim_{n \rightarrow \infty} (A \rightarrow B_n) = A \rightarrow B$. In fact, it follows from $\lim_{n \rightarrow \infty} B_n = B$ that $\lim_{n \rightarrow \infty} \neg B_n = \neg B$, hence it follows from Lemma 4 that

$$\lim_{n \rightarrow \infty} (A \rightarrow B_n) \approx \lim_{n \rightarrow \infty} (\neg B_n \rightarrow \neg A) = \neg B \rightarrow \neg A \approx A \rightarrow B$$

where \approx means logically equivalent. Suppose now $\lim_{n \rightarrow \infty} (A_n, B_n) = (A, B)$, i.e., $\lim_{n \rightarrow \infty} A_n = A$, and $\lim_{n \rightarrow \infty} B_n = B$, and $\varepsilon \in (0, 1)$. It follows from Lemma 4 that there exists M_1 large enough such that

$$\rho(A_n \rightarrow B_m, A \rightarrow B_m) < 1 - (1 - \varepsilon)^2$$

holds for every $B_m \in F(S)$ whenever $m \geq M_1$. Similarly, there exists M_2 large enough such that

$$\rho(A \rightarrow B_m, A \rightarrow B) < 1 - (1 - \varepsilon)^2$$

holds whenever $m \geq M_2$. Let $M = \max\{M_1, M_2\}$, then

$$\begin{aligned} \rho(A_k \rightarrow B_k, A \rightarrow B) &\leq \rho(A_k \rightarrow B_k, A_k \rightarrow B) + \rho(A_k \rightarrow B, A \rightarrow B) \\ &\leq 2(1 - (1 - \varepsilon)^2) \leq 2\varepsilon(2 - \varepsilon) \end{aligned} \quad (31)$$

holds whenever $k \geq M$. Since ε is arbitrary we have from (31) that $\lim_{n \rightarrow \infty} \rho(A_k \rightarrow B_k, A \rightarrow B) = 0$, or $\lim_{n \rightarrow \infty} (A_n \rightarrow B_n) = A \rightarrow B$. This completes the proof. \square

Remark 1. We proved in [10] that the Lukasiewicz implication operator \rightarrow is also continuous in the R_L -type fuzzy logic metric space $(F(S), \rho_L)$. From this fact and Theorem 2 a question naturally arises: does every implication operator adjointed by an LC- t -norm possess the above mentioned continuity? If the answer is positive, then how can we prove it? This is an interesting question.

8. An algorithm for fuzzy reasoning and its application

Let U, V be two universes of discourse, the sets consisting of all fuzzy subsets of U and V will be denoted by $F(U)$ and $F(V)$ respectively. FMP is the basic form of fuzzy reasoning which can be stated as follows:

$$\begin{array}{ll} \text{suppose that} & \widetilde{A}(u) \rightarrow \widetilde{B}(v) \\ \text{and given} & \widetilde{A}^*(u) \\ \hline \text{calculate} & \widetilde{B}^*(v) \end{array} \quad u \in U, v \in V, \quad (32)$$

where $\widetilde{A}, \widetilde{A}^* \in F(U)$, $\widetilde{B}, \widetilde{B}^* \in F(V)$. This question can be formalized to be the following question in L^* :

$$\begin{array}{ll} \text{suppose that} & A \rightarrow B \\ \text{and given} & A^* \\ \hline \text{calculate} & B^* \end{array} \quad (33)$$

or in logic words: our task is to find out a suitable conclusion B^* of the prerequisites $A \rightarrow B$ and A^* . We call this question generalized modus ponens (briefly, GMP). As the conclusion of $A \rightarrow B$ and A^* , B^* certainly should fulfill the condition that

$$\{A \rightarrow B, A^*\} \vdash B^*. \quad (34)$$

But there are too many B^* satisfying (34), for example, if B^* is a tautology (theorem), then B^* satisfies (34), and it is clear not what we need because a tautology B^* is a conclusion of any given set Γ of prerequisites, i.e., $\Gamma \vdash B^*$ is always true no matter what Γ is. Hence tautologies are not suitable conclusions of (34). Recall that there exists an intrinsic pre-order \leq on $F(S)$ such that $\forall A, B \in F(S)$, $A \leq B$ if and only if $\vdash A \rightarrow B$, or $A \rightarrow B$ is a tautology (see [8]), hence tautologies are the largest members in $F(S)$, and it is reasonable to ask B^* in (33) to be as small as possible. Precisely speaking, the conclusion B^* of GMP (33) should satisfy the following two conditions:

- (i) B^* satisfies (34)
- (ii) if C satisfies (34), then $B^* \leq C$ ($C \in F(S)$).

Theorem 3. Let $B^* = (A^*)^2 \otimes (A \rightarrow B)^2$, then B^* satisfies conditions (i) and (ii).

Proof. Recall that $A \otimes B = \neg(A \rightarrow \neg B)$, and $A^2 = A \otimes A$, the Deduction Theorem of L^* says that

$$\Gamma \cup \{A\} \vdash B \quad \text{if and only if} \quad \Gamma \vdash A^2 \rightarrow B, \quad \Gamma \subset F(S), \quad A, B \in F(S). \quad (35)$$

Moreover, it is well-known in L^* that

$$A \rightarrow (B \rightarrow C) \approx A \otimes B \rightarrow C, \quad A, B, C \in F(S), \quad (36)$$

where \approx is the logical equivalence relation (see [2]). Since

$$\vdash (A^*)^2 \otimes (A \rightarrow B)^2 \rightarrow B^*, \quad \text{where } B^* = (A^*)^2 \otimes (A \rightarrow B)^2 \quad (37)$$

is clearly true, it follows from (36) and (35) that the following assertions are true

$$\vdash (A^*)^2 \rightarrow ((A \rightarrow B)^2 \rightarrow B^*),$$

$$\{A^*\} \vdash (A \rightarrow B)^2 \rightarrow B^*,$$

$$\{A^*, A \rightarrow B\} \vdash B^*.$$

Hence B^* satisfies condition (i). Secondly, assume that C satisfies (34), i.e.,

$$\{A^*, A \rightarrow B\} \vdash C, \quad C \in F(S), \quad (38)$$

then it follows from the Deduction Theorem of L^* that

$$\{A^*\} \vdash (A \rightarrow B)^2 \rightarrow C,$$

$$\vdash (A^*)^2 \rightarrow ((A \rightarrow B)^2 \rightarrow C). \quad (39)$$

Therefore it follows from (39) and (36) that

$$\vdash (A^*)^2 \otimes (A \rightarrow B)^2 \rightarrow C,$$

and this is exactly $\vdash B^* \rightarrow C$, hence $B^* \leq C$, i.e., B^* satisfies condition (ii). The proof of Theorem 3 is completed. \square

Theorem 3 tells us that the solution B^* of GMP (33) is the smallest formula satisfying

$$(A^*)^2 \otimes (A \rightarrow B)^2 \leq B^*. \quad (40)$$

This hints at an algorithm for solving the question of FMP, i.e., \tilde{B}^* in (32) should satisfy the following conditions:

(i) $\forall u \in U, \forall v \in V$,

$$(\tilde{A}^*(u))^2 \otimes (\tilde{A}(u) \rightarrow \tilde{B}(v))^2 \leq \tilde{B}^*(v), \quad u \in U, v \in V.$$

(ii) $\forall v \in V, \tilde{B}^*(v)$ takes the smallest value when u varies in U .

Therefore we have

$$\tilde{B}^*(v) = \sup_{u \in U} \{(\tilde{A}^*(u))^2 \otimes (\tilde{A}(u) \rightarrow \tilde{B}(v))^2\}, \quad v \in V. \quad (41)$$

Now $\tilde{A}^*(u)$, $\tilde{A}(u) \rightarrow \tilde{B}(v)$, and $\tilde{B}^*(v)$ are all real numbers in $[0, 1]$, and we have from (41) and (12) that $(\tilde{A}^*(u))^2 = 0$ and $(\tilde{A}(u) \rightarrow \tilde{B}(v))^2 = 0$ if $\tilde{A}^*(u) \leq \frac{1}{2}$ and $\tilde{A}(u) \rightarrow \tilde{B}(v) \leq \frac{1}{2}$ respectively. Therefore we have the following algorithm for computing the conclusion \tilde{B}^* of FMP (32):

Algorithm. The conclusion \tilde{B}^* of FMP (32) can be computed as follows:

$$\tilde{B}^*(v) = \sup_{u \in U} \left\{ \tilde{A}^*(u) \wedge (\tilde{A}(u) \rightarrow \tilde{B}(v)) \mid \tilde{A}^*(u) > \frac{1}{2}, \tilde{A}(u) \rightarrow \tilde{B}(v) > \frac{1}{2} \right\}, \quad v \in V. \quad (42)$$

Proof. It follows from (12) that

$$\begin{aligned} (\tilde{A}^*(u))^2 \otimes (\tilde{A}(u) \rightarrow \tilde{B}(v))^2 &= \begin{cases} \tilde{A}^*(u) \otimes (\tilde{A}(u) \rightarrow \tilde{B}(v)), & \tilde{A}^*(u) > \frac{1}{2} \text{ and } (\tilde{A}(u) \rightarrow \tilde{B}(v)) > \frac{1}{2}, \\ 0, & \tilde{A}^*(u) \leq \frac{1}{2} \text{ or } (\tilde{A}(u) \rightarrow \tilde{B}(v)) \leq \frac{1}{2} \end{cases} \\ &= \begin{cases} \tilde{A}^*(u) \wedge (\tilde{A}(u) \rightarrow \tilde{B}(v)), & \tilde{A}^*(u) > \frac{1}{2} \text{ and } (\tilde{A}(u) \rightarrow \tilde{B}(v)) > \frac{1}{2}, \\ 0, & \tilde{A}^*(u) \leq \frac{1}{2} \text{ or } (\tilde{A}(u) \rightarrow \tilde{B}(v)) \leq \frac{1}{2}. \end{cases} \quad \square \end{aligned}$$

Hence it follows from (41) that (42) holds.

Remark 2. The pioneering form of the FMP conclusion \tilde{B}^* calculated by using Zadeh's CRI (Compositional Rule of Inference) method is as follows (see [23,24]):

$$\tilde{B}^*(v) = \sup_{u \in U} \{ \tilde{A}^*(u) \wedge (\tilde{A}(u) \rightarrow \tilde{B}(v)) \}, \quad v \in V. \quad (43)$$

Comparing (43) with (42) we see that, when computing the supremum, every value of $\tilde{A}^*(u)$ and $\tilde{A}(u) \rightarrow \tilde{B}(v)$ is considered in (43), while the small values of $\tilde{A}^*(u)$ and $\tilde{A}(u) \rightarrow \tilde{B}(v)$ are not taken into account in (42). It seems that (42) is not remarkably different from (43) because small values are not important for computing the supremum. However, (42) seems to have the following advantages:

- (i) (42) has a formal logic background.
- (ii) sometimes (42) is easier to be computed than (43).

Example 1. Suppose that $U = V = [0, 1]$, $\tilde{A}, \tilde{A}^* \in \mathbf{F}(U)$, $\tilde{B} \in \mathbf{F}(V)$ are as follows:

$$\tilde{A}(u) = \begin{cases} 2u, & 0 \leq u \leq \frac{1}{2}, \\ 2u - 1, & \frac{1}{2} < u \leq 1. \end{cases}$$

$$\tilde{A}^*(u) = u.$$

$$\tilde{B}(v) = \frac{1}{2}(v + 1). \quad u, v \in [0, 1].$$

Compute the FMP Conclusion $\tilde{B}^* \in \mathbf{F}(V)$ by using the CRI method, where the implication operator \rightarrow is R_0 .

Solution. It follows from (43) that

$$\begin{aligned} \tilde{B}^*(v) &= \sup_{u \in [0, 1]} \{ \tilde{A}^*(u) \wedge (\tilde{A}(u) \rightarrow \tilde{B}(v)) \} \\ &= \sup_{u \in [0, \frac{1}{2}]} \left\{ u \wedge \left(2u \rightarrow \frac{1}{2}(v + 1) \right) \right\} \vee \sup_{u \in (\frac{1}{2}, 1]} \left\{ u \wedge \left((2u - 1) \rightarrow \frac{1}{2}(v + 1) \right) \right\}. \end{aligned} \quad (44)$$

The first part of the RHS of (44) equals

$$\begin{aligned} \sup_{u \in [0, \frac{1}{2}]} \left\{ u \wedge \left(2u \rightarrow \frac{1}{2}(v + 1) \right) \right\} &= \sup \left\{ u \wedge \left(2u \rightarrow \frac{1}{2}(v + 1) \right) \mid 0 \leq u \leq \frac{1}{4}(v + 1) \right\} \\ &\quad \vee \sup \left\{ u \wedge \left(2u \rightarrow \frac{1}{2}(v + 1) \right) \mid \frac{1}{4}(v + 1) < u \leq \frac{1}{2} \right\} \\ &= \sup \left\{ u \mid 0 \leq u \leq \frac{1}{4}(v + 1) \right\} \end{aligned}$$

$$\begin{aligned}
& \vee \sup \left\{ u \wedge \left[(1 - 2u) \vee \frac{1}{2}(v + 1) \right] \middle| \frac{1}{4}(v + 1) < u \leq \frac{1}{2} \right\} \\
&= \frac{1}{4}(v + 1) \vee \sup \left\{ u \wedge \frac{1}{2}(v + 1) \middle| \frac{1}{4}(v + 1) < u \leq \frac{1}{2} \right\} \\
&= \frac{1}{4}(v + 1) \vee \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

The second part of the RHS of (44) equals

$$\begin{aligned}
& \sup \left\{ u \wedge \left((2u - 1) \rightarrow \frac{1}{2}(v + 1) \right) \middle| \frac{1}{2} < u \leq \frac{1}{4}(v + 3) \right\} \\
& \vee \sup \left\{ u \wedge \left((2u - 1) \rightarrow \frac{1}{2}(v + 1) \right) \middle| \frac{1}{4}(v + 3) < u \leq 1 \right\}. \quad (45)
\end{aligned}$$

Notice that $u \leq \frac{1}{4}(v + 3)$ implies that $2u - 1 \leq \frac{1}{2}(v + 1)$ it follows from (12) that the expression (45) equals

$$\begin{aligned}
& \sup \left\{ u \middle| \frac{1}{2} < u \leq \frac{1}{4}(v + 3) \right\} \vee \sup \left\{ u \wedge \left[(1 - (2u - 1)) \vee \frac{1}{2}(v + 1) \right] \middle| \frac{1}{4}(v + 3) < u \leq 1 \right\} \\
&= \frac{1}{4}(v + 3) \vee \sup \left\{ u \wedge 2(1 - u) \middle| \frac{1}{4}(v + 3) < u \leq 1 \right\} \vee \sup \left\{ u \wedge \frac{1}{2}(v + 1) \middle| \frac{1}{4}(v + 3) < u \leq 1 \right\}. \quad (46)
\end{aligned}$$

It can be verified that $\sup \{ u \wedge 2(1 - u) \mid \frac{1}{4}(v + 3) < u \leq 1 \} = \frac{1}{2}(v + 1)$, and $\frac{1}{4}(v + 3) \geq \frac{1}{2}(v + 1)$ holds whenever $v \leq 1$, hence the expression (46) equals $\frac{1}{4}(v + 3)$, and finally we have from (44) that $\tilde{B}^*(v) = \frac{1}{4}(v + 3)$.

Example 2. Suppose that U, V, A, \tilde{A}^* , and \tilde{B} are the same as in Example 1, compute the FMP conclusion \tilde{B}^* by using the algorithm (42).

Solution. Since $a \rightarrow b \geq b$, it follows from $\tilde{B}(v) = \frac{1}{2}(v + 1) > \frac{1}{2}$ whenever $v \neq 0$ that $\tilde{A}(u) \rightarrow \tilde{B}(v) > \frac{1}{2}$ ($v \neq 0$). Therefore we have from (42) that

$$\tilde{B}^*(v) = \sup_{u \in (\frac{1}{2}, 1]} \left\{ u \wedge ((2u - 1) \rightarrow \frac{1}{2}(v + 1)) \right\} = \frac{1}{4}(v + 3), \quad v \neq 0. \quad (47)$$

Moreover, it follows from (42) that

$$\tilde{B}^*(0) = \sup_{u \in (\frac{1}{2}, 1]} \left\{ u \wedge \left((2u - 1) \rightarrow \frac{1}{2} \right) \middle| (2u - 1) \rightarrow \frac{1}{2} > \frac{1}{2} \right\} = \frac{3}{4} = \frac{1}{4}(0 + 3).$$

Hence $\tilde{B}^*(v) = \frac{1}{4}(v + 3)$.

The above examples show that sometimes (42) is easier to be computed than (43). But the most important thing is that (42) has a solid logic background.

9. Root of a theory in $F(S)$

Now that we have the R_0 -type fuzzy pseudo-metric ρ on the set $F(S)$ of all propositions, it is possible to discuss approximate reasoning in $(F(S), \rho)$. As is well-known, a subset Γ of $F(S)$ is often called a *Theory* in $F(S)$. Starting from Γ and axioms, propositions obtained by using MP in finite steps are called Γ -conclusions. Let $D(\Gamma)$ be the set consisting of all Γ -conclusions, then

$$D(\Gamma) = \{A \in F(S) \mid \Gamma \vdash A\}.$$

Notice that different theories may have same conclusions. For example, let $\Gamma_1 = \{A, A \rightarrow B, B \rightarrow C\}$ and $\Gamma_2 = \{A, B, C\}$, then it is easy to prove that $D(\Gamma_1) = D(\Gamma_2)$, where Γ_2 is clearly simpler than Γ_1 . An interesting

question is: Let Γ be a theory, if or not we can find a smallest theory Γ_0 containing only one proposition, say, A , satisfying the following conditions:

(Rt. i) $A \in D(\Gamma)$, i.e., A is a Γ -conclusion.

(Rt. ii) $\forall B \in D(\Gamma), \vdash A \rightarrow B$ holds.

If such an A exists, then we call A a root of Γ , and denote it by $A = \text{root}(\Gamma)$.

Corollary 1. *If A and B are roots of Γ , then $A \sim B$, i.e., whenever Γ 's root exists, it is unique in the sense of provable equivalence.*

Proof. Suppose that both A and B are roots of Γ , then it follows from condition (Rt. i) that $A, B \in D(\Gamma)$, and then it follows from condition (Rt. ii) that both $\vdash A \rightarrow B$ and $\vdash B \rightarrow A$ hold. Hence $A \sim B$. \square

The following corollary follows from [10] and Corollary 1 immediately.

Corollary 2. *Let A and B be roots of Γ , then $\rho(A, B) = 0$ in $(F(S), \rho)$ and hence $\rho(A, C) = \rho(B, C)$ holds for every formula $C \in F(S)$.*

Notice that a theory Γ may have many Γ -conclusions, e.g., every theorem is a Γ -conclusion. It is clear that among all Γ -conclusions the root A of Γ (if it exists) is, in a sense, the best Γ -conclusion, because all Γ -conclusions can be deduced from A .

Now let us re-consider the deduction rule MP, it says that B can be obtained from $A \rightarrow B$ and A . In other words, B is a Γ -conclusion where $\Gamma = \{A \rightarrow B, A\}$. Notice that B is not the best Γ -conclusion because $B \neq \text{root}(\Gamma)$. In fact, let $E = A^2 \otimes (A \rightarrow B)^2$, then

(i) Since in the system L^* (see [8])

$$G \rightarrow (H \rightarrow K) \sim G \otimes H \rightarrow K,$$

hence the deduction theorem (13) can be written as

$$\{A, B\} \vdash C \quad \text{if and only if} \quad \vdash A^2 \otimes B^2 \rightarrow C. \quad (48)$$

Therefore we have from $\vdash A^2 \otimes (A \rightarrow B)^2 \rightarrow A^2 \otimes (A \rightarrow B)^2$ that

$$\{A, A \rightarrow B\} \vdash A^2 \otimes (A \rightarrow B)^2.$$

(ii) Assume that $\{A \rightarrow B, A\} \vdash C$, then we have from (48) (substitute B by $A \rightarrow B$) that $\vdash A^2 \otimes (A \rightarrow B)^2 \rightarrow C$, hence $C \in D(\{A^2 \otimes (A \rightarrow B)^2\})$ and therefore we see that $E = A^2 \otimes (A \rightarrow B)^2$ is a root of $\{A \rightarrow B, A\}$ and $B \neq E$.

Notice also that even if B is not root of $\Gamma = \{A \rightarrow B, A\}$ it is the most important Γ -conclusion because it extricates itself from the complex form $A \rightarrow B$ where A is involved. But the concept of root of a theory Γ plays a key role when approximate reasoning is discussed. In fact, before a certain conclusion of a theory Γ is specified, questions like

“How far is the distance between A and a Γ -conclusion?” do not make sense, because Γ may have many conclusions, and the value of $\rho(A, B)$ varies when B changes in $D(\Gamma)$. Now let us turn back to the GMP question expressed by (33). Theorem 3 gives the answer to this question, and it can be verified that $B^* = (A^*)^2 \otimes (A \rightarrow B)^2$ is a root of $\Gamma^* = \{A \rightarrow B, A^*\}$. Suppose that “ A^* is close to A ” in $(F(S), \rho)$, requiring under this condition that “ B^* is close to B ” is not reasonable, because where B^* is a root of Γ^* , B is not a root of $\Gamma = \{A \rightarrow B, A\}$. Nevertheless, the following question is reasonable:

Question. If or not the root B^* of Γ^* is close to the root of Γ when A^* is close to A ?

The following theorem answers this question positively.

Theorem 4. *Let $\Gamma = \{A \rightarrow B, A\}$ and $\Gamma^* = \{A \rightarrow B, A^*\}$ be two theories in the R_0 -type fuzzy pseudo-metric space $(F(S), \rho)$, and $\varepsilon > 0$. If $\rho(A^*, A) < \frac{\varepsilon}{16}$, then*

$$\rho(\text{root}(\Gamma^*), \text{root}(\Gamma)) < \varepsilon.$$

Some materials obtained in [16] are needed for the proof of Theorem 4, we re-write them as the following lemma.

Lemma 5. Let $(F(S), \rho)$ be the R_0 -type fuzzy pseudo-metric space, $A, B, C \in F(S)$, $\tau(A) = \tau_R(A)$ is defined by (16) where $R = R_0$.

- (i) If $\vdash A \rightarrow B$, then $\tau(A) \leq \tau(B)$.
- (ii) If $\tau(A \rightarrow B) \geq \alpha$, $\tau(B \rightarrow C) \geq \beta$, then $\tau(A \rightarrow C) \geq \alpha + \beta - 1$.
- (iii) If $\tau(A) \geq 1 - \delta$, $\tau(B) > 1 - \delta$, then $\tau(A \wedge B) > (1 - \sqrt{2\delta})^2$.

Proof of Theorem 4. The following are a list of well-known theorems in the system L^* :

$$\vdash (A \rightarrow B) \wedge (B \rightarrow A) \rightarrow (A \rightarrow B). \quad (49)$$

$$\vdash (A \rightarrow B) \rightarrow ((B \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)). \quad (50)$$

$$\vdash (\neg B \rightarrow \neg A) \rightarrow ((A \rightarrow \neg B) \rightarrow (A \rightarrow \neg A)). \quad (51)$$

We see from (1), (2) and (16) that

$$\rho(A, B) = 1 - \xi(A, B) = 1 - \tau((A \rightarrow B) \wedge (B \rightarrow A)), \quad (52)$$

where the subscript $R = R_0$ in ρ_R , ξ_R and τ_R is omitted.

Suppose that $\rho(A, B) < \delta$, then we have from (52) that $\tau((A \rightarrow B) \wedge (B \rightarrow A)) > 1 - \delta$, and hence it follows from Lemma 5(i), (49)–(51) and $\tau(A \rightarrow B) = \tau(\neg B \rightarrow \neg A)$ that

$$\begin{aligned} \tau(A \rightarrow B) &> 1 - \delta, \\ \tau((B \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)) &> 1 - \delta, \end{aligned} \quad (53)$$

$$\tau((A \rightarrow \neg B) \rightarrow (A \rightarrow \neg A)) > 1 - \delta. \quad (54)$$

It follows from (53) and (54), and Lemma 5(ii) that

$$\tau((B \rightarrow \neg B) \rightarrow (A \rightarrow \neg A)) > (1 - \delta) + (1 - \delta) - 1 = 1 - 2\delta. \quad (55)$$

Notice that

$$A^2 \rightarrow B^2 = \neg(A \rightarrow \neg A) \rightarrow \neg(B \rightarrow \neg B) = (B \rightarrow \neg B) \rightarrow (A \rightarrow \neg A).$$

We have from (55) that $\tau(A^2 \rightarrow B^2) > 1 - 2\delta$. Similarly, we can prove from $\rho(A, B) < \delta$ that $\tau(B^2 \rightarrow A^2) > 1 - 2\delta$. Moreover, we have from

$$\begin{aligned} A^2 \otimes C \rightarrow B^2 \otimes C &= \neg(A^2 \rightarrow \neg C) \rightarrow \neg(B^2 \rightarrow \neg C) \sim (B^2 \rightarrow \neg C) \rightarrow (A^2 \rightarrow \neg C), \\ \vdash (A^2 \rightarrow B^2) &\rightarrow ((B^2 \rightarrow C) \rightarrow (A^2 \rightarrow C)), \end{aligned}$$

and $\tau(A^2 \rightarrow B^2) > 1 - 2\delta$ that

$$\tau(A^2 \otimes C \rightarrow B^2 \otimes C) > 1 - 2\delta. \quad (56)$$

Similarly, we also have that

$$\tau(B^2 \otimes C \rightarrow A^2 \otimes C) > 1 - 2\delta. \quad (57)$$

Hence it follows from (56), (57), and Lemma 5(iii) that

$$\tau((A^2 \otimes C \rightarrow B^2 \otimes C) \wedge (B^2 \otimes C \rightarrow A^2 \otimes C)) > (1 - 2\sqrt{\delta})^2 > 1 - 4\sqrt{\delta}. \quad (58)$$

Suppose that $B = A^*$ and $C = (A \rightarrow B)^2$, then $\rho(A, B) < \delta = \frac{\varepsilon}{16}$ and we have from (58) and (2) that

$$\rho((A^*) \otimes (A \rightarrow B)^2, A^2 \otimes (A \rightarrow B)^2) < 4\sqrt{\delta} = \varepsilon,$$

i.e., $\rho(\text{root}(\Gamma^*), \text{root}(\Gamma)) < \varepsilon$. This proves Theorem 4. \square

10. Conclusion

For reducing the gap between fuzzy logic and AI, a formalized version of FMP, i.e., GMP, is proposed and solved in the fuzzy logic system L^* . Then the corresponding numerical algorithm with rigorous logic foundation for solving FMP is proposed, and this algorithm has the benefit that it can be easily computed because only a part of the variables of the universe are taken into account for computing related supremums. As a preparation, advantages of employing fuzzy pseudo-metric and the R_0 -implication operator are clarified. And virtues of the fuzzy pseudo-metric space $(F(S), \rho)$ are also analyzed. Based on the concept of root of a logic theory, an elementary approximate property of GMP is discussed, and a systematic approximate reasoning theory can be established from there which will be investigated subsequently.

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